

# POSITIVE RATIONAL NODAL LEAVES ON SURFACES

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**ABSTRACT.** We consider singular holomorphic foliations on compact complex surfaces with invariant rational nodal curve of positive self-intersection. Then, under some assumptions, we list all possible foliations.

## 1. INTRODUCTION

Let  $X$  be a compact complex surface and  $\mathcal{F}$  a codimension one singular holomorphic foliation on it. This work aims at generalizing the following result of Brunella (see [2] and [3]):

**Theorem 1.1.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$  and let  $C \subset X$  be a rational curve with a node  $p$ , invariant by  $\mathcal{F}$ , and with  $C^2 = 3$ . Suppose that  $p$  is a reduced nondegenerate singularity of  $\mathcal{F}$ , and that it is the unique singularity of  $\mathcal{F}$  on  $C$ . Then  $\mathcal{F}$  is unique up to birational transformations.*

The unique foliation given by Theorem 1.1 will be called *Brunella's very special foliation* (see subsection 3.1 for the definition).

But, what occurs if  $C^2$  is an arbitrary positive integer? More specifically, we want to study/classify foliations on compact complex surfaces satisfying assumptions similar to the ones of Theorem 1.1 with the hypothesis  $C^2 = 3$  replaced by  $C^2 = n$ , where  $n$  is an arbitrary positive integer.

**Definition 1.2.** Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$ . A *link* for  $\mathcal{F}$  is a rational nodal curve  $C \subset X$  with only one node  $p \in C$  such that:

- (1)  $C$  is *positive*, that is,  $C^2 = n > 0$ ;
- (2)  $C$  is  $\mathcal{F}$ -invariant;
- (3)  $p$  is a reduced nondegenerate singularity of  $\mathcal{F}$ , and it is the unique singularity of  $\mathcal{F}$  on  $C$ .

The existence of  $C \subset X$ ,  $C^2 = n > 0$ , implies that  $X$  is a projective surface (see [1], Theorem 6.2, page 160).

Our main purpose in this paper is to prove the following theorem:

**Theorem 1.3.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$  and let  $C \subset X$  be a link for  $\mathcal{F}$ . Then we have only three possibilities, each one unique up to birational transformations:*

- (1)  $C^2 = 1$  and  $\mathcal{F}$  is birational to a foliation  $\mathcal{F}_1$  on  $Bl_3(\mathbb{P}^2)/\alpha$ , where  $\alpha \in Aut(Bl_3(\mathbb{P}^2))$  and  $Bl_3(\mathbb{P}^2)$  is a blow-up of  $\mathbb{P}^2$  in three non-collinear points;
- (2)  $C^2 = 2$  and  $\mathcal{F}$  is birational to a foliation  $\mathcal{F}_2$  on  $\mathbb{P}^1 \times \mathbb{P}^1/\beta$ ,  $\beta \in Aut(\mathbb{P}^1 \times \mathbb{P}^1)$ ;

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- (3)  $C^2 = 3$  and  $\mathcal{F}$  is birational to a foliation  $\mathcal{F}_3$  on  $\mathbb{P}^2/\gamma$  (Brunella's very special foliation),  $\gamma \in \text{Aut}(\mathbb{P}^2)$ .

## 2. SOME RESULTS IN ALGEBRAIC AND COMPLEX GEOMETRY

For the reader's convenience, we summarize here some classical fundamentals results which will be used along this paper.

### 2.1. Bimeromorphic geometry.

**Definition 2.1** (*Exceptional Curves*). A compact, reduced, connected curve  $C$  on a nonsingular surface  $X$  is called *exceptional*, if there is a bimeromorphic map  $\pi : X \rightarrow Y$  such that  $C$  is exceptional for  $\pi$ , i.e., if there is an open neighbourhood  $U$  of  $C$  in  $X$ , a point  $y \in Y$ , and a neighbourhood  $V$  of  $y$  in  $Y$ , such that  $\pi$  maps  $U - C$  biholomorphically onto  $V - \{y\}$ , whereas  $\pi(C) = y$ . We shall express this situation also by saying that  $C$  is *contracted* to  $y$ .

**Theorem 2.2** (Grauert's criterion, [1], page 91). *A reduced, compact connected curve  $C$  with irreducible components  $C_i$  on a smooth surface is exceptional if and only if the intersection matrix  $(C_i \cdot C_j)$  is negative definite.*

**Definition 2.3** (*Exceptional curves of the first kind*). These are nonsingular rational curves with self-intersection  $-1$ . Frequently we call such curves  $(-1)$ -curves. A very useful characterisation of  $(-1)$ -curves is given by

**Theorem 2.4** ([1], page 97). *Let  $X$  be a nonsingular surface,  $E \subset X$  a  $(-1)$ -curve and  $\pi : X \rightarrow Y$  the map contracting  $E$ . Then  $y = \pi(E)$  is nonsingular on  $Y$ .*

**Theorem 2.5** (Uniqueness of the  $\sigma$ -process, [1], page 98). *Let  $X$  and  $Y$  be smooth surfaces and  $\pi : X \rightarrow Y$  a bimeromorphic map. If  $E = \pi^{-1}(y)$  is an irreducible curve, then near  $E$ , the map  $\pi$  is equivalent to the  $\sigma$ -process with centre  $y$ .*

**Lemma 2.6** (Factorization lemma, [1], page 98). *Let  $\pi : X \rightarrow Y$  be a bimeromorphic map with  $X, Y$  nonsingular surfaces. Unless it is an isomorphism, there is a factorization  $\pi = \pi' \circ \sigma$ , where  $\sigma : X \rightarrow X$  is a  $\sigma$ -process.*

**Corollary 2.7** (Decomposition of bimeromorphic maps, [1], page 98). *Let  $X, Y$  be non-singular and  $\pi : X \rightarrow Y$  a bimeromorphic map. Then  $\pi$  is equivalent to a succession of  $\sigma$ -transforms, which locally (with respect to  $Y$ ) are finite in number.*

**Theorem 2.8** ([1], page 192). *Let  $X$  be a compact surface and  $C$  a smooth rational curve on  $X$ . If  $C^2 = 0$ , then there exists a modification  $\pi : X \rightarrow Y$ , where  $Y$  is ruled, such that  $C$  meets no exceptional curve of  $\pi$ , and  $\pi(C)$  is a fibre of  $\pi$ .*

### 2.2. Complex geometry.

**Lemma 2.9** ([16], Lemma 5). *Let  $X$  be a compact complex manifold of dimension  $n > 1$ ,  $K$  a compact subset of  $X$  and  $E$  a holomorphic vector bundle over  $X$ . If  $X$  is strongly pseudoconvex, then every section  $s$  of  $E$  over  $X - K$  can be extended to a meromorphic section  $\tilde{s}$  over all of  $X$ .*

**Lemma 2.10** ([11], page 32). *Let  $X$  be a compact complex surface and  $C \subset X$  a compact irreducible curve. If  $C^2 > 0$  then  $X - C$  is strongly pseudoconvex.*

## 3. EXISTENCE

For us a *cycle of smooth rational curves* (or simple a *cycle*) always means the union of a finite number of smooth rational curves in general position  $C_i$ ,  $i = 1, \dots, m$ ,  $m > 1$ , such that: if  $m = 2$ , then  $\#C_1 \cap C_2 = 2$ ; if  $m > 2$ , then  $\#C_i \cap C_{(i+1)} = \#C_1 \cap C_m = 1$ ,  $i = 1, \dots, m-1$ , otherwise  $\#C_i \cap C_j = 0$ .

**3.1. Existence for  $C^2 = 3$  (Brunella's very special foliation).** Let  $\mathcal{L}$  be the linear foliation on  $\mathbb{P}^2$  given in affine coordinates by the linear 1-form

$$\omega = \lambda y dx - x dy = \left( \frac{1 \pm \sqrt{-3}}{2} \right) y dx - x dy.$$

This foliation has an invariant cycle of three lines  $C_1 \cup C_2 \cup C_3$ . Moreover, the foliation  $\mathcal{L}$  is  $\gamma$ -invariant, where  $\gamma : (s : t : u) \mapsto (u : s : t)$  is in  $\text{Aut}(\mathbb{P}^2)$ .

The quotient foliation  $\mathcal{F}_3 = \mathcal{L}/\gamma$  obtained by taking the quotient of  $(\mathbb{P}^2, \mathcal{L})$  by the group generated by  $\gamma$  is, by definition, *Brunella's very special foliation*.

Note that the choose of  $\lambda$  don't affect the birational class of  $\mathcal{F}_3$ , since the involution  $(x, y) \mapsto (y, x)$  conjugates the two possible constructions.

**3.2. Existence for  $C^2 = 2$ .** We take the foliation  $\mathcal{M}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  given in affine coordinates  $(x, y)$  by the linear 1-form

$$\omega = \lambda y dx - x dy = \pm \sqrt{-1} y dx - x dy.$$

where  $\lambda = \pm \sqrt{-1}$ . Then it leaves invariant the cycle of four lines

$$(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1),$$

in which the only singularities are the crossing points, each one reduced nondegenerate. The automorphism of order 4

$$\beta : (u : v, z : w) \mapsto (z : w, v : u).$$

is such that, in affine coordinates  $(x, y)$ ,  $\beta(x, y) = (y, \frac{1}{x})$  and

$$\beta^* \omega = \beta^* (\lambda y dx - x dy) = \lambda \frac{1}{x} dy - y \left( -\frac{1}{x^2} \right) dx,$$

hence, since  $\lambda = \pm \sqrt{-1}$ ,

$$\omega \wedge \beta^* \omega = (\lambda y dx - x dy) \wedge \left( \lambda \frac{1}{x} dy + \frac{y}{x^2} dx \right) = (\lambda^2 + 1) \frac{y}{x} dx \wedge dy = 0.$$

Note that  $\beta$  permutes cyclically the cycle of four lines

$$(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1).$$

Then the quotient foliation  $\mathcal{F}_2$  obtained by taking the quotient of  $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{M})$  by the group generated by  $\beta$  is the desired foliation, that is,  $\mathcal{F}_2$  has a link of self-intersection 2.

Again the choose of  $\lambda$  don't affect the birational class of  $\mathcal{F}_2$ , since the involution  $(u : v, z : w) \mapsto (z : w, u : v)$  conjugates the two possible constructions.

**3.3. Existence for  $C^2 = 1$ .** Let  $\mathcal{L}$  and  $\gamma$  as in subsection 3.1. Recall that  $\mathcal{L}$  has a cycle of three invariant lines  $C_1 \cup C_2 \cup C_3$ , where  $C_i = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2 \mid z_i = 0\}$ ,  $i = 1, 2, 3$ . Consider the standard Cremona transformation  $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ ,  $f([z_1 : z_2 : z_3]) = [z_2 z_3 : z_1 z_3 : z_1 z_2]$ . Note that  $\mathcal{L}$  is  $f$ -invariant.

If we blow-up the crossing points of the cycle of three  $\mathcal{L}$ -invariant projective lines  $C_1 \cup C_2 \cup C_3$ , we obtain a birational morphism  $\pi_3 : Bl_3(\mathbb{P}^2) \rightarrow \mathbb{P}^2$  and a foliation  $\mathcal{N} = \pi_3^* \mathcal{L}$  with an invariant cycle of six smooth rational  $(-1)$ -curves, say  $\tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3 \cup C_4 \cup C_5 \cup C_6$ , in which the singularities of  $\mathcal{N}$  are only the crossing points (and they are reduced nondegenerate). Note that  $\alpha = \pi_3^{-1} \circ f \circ \pi_3 : Bl_3(\mathbb{P}^2) \rightarrow Bl_3(\mathbb{P}^2)$  becomes an automorphism of order six that preserves the foliation and permutes cyclically the cycle of six invariant rational curves.

The quotient foliation  $\mathcal{F}_1 = \mathcal{N}/\alpha$  has a link of self-intersection 1, hence  $\mathcal{F}_1$  is the desired foliation.

#### 4. RICCATI FOLIATIONS

We develop here the first tools to proof our main result.

Let  $\mathcal{F}$  be a foliation on  $X$  which is Riccati with respect to a fibration  $\pi : X \rightarrow B$ , where  $B$  is a nonsingular curve. If  $R$  is a regular fibre of  $\pi$  which is  $\mathcal{F}$ -invariant, then ([2, Chapter 4]): there are at most two singularities on  $R$  and there exists coordinates  $(x, y) \in D \times \mathbb{P}^1$  around  $R$ , where  $D$  is a disc, such that the foliation is given by the 1-form

$$\omega = (a(x)y^2 + b(x)y + c(x))dx + h(x)dy.$$

Let  $q$  be a singularity for  $\omega$ . After a change in the  $y$  coordinate, we can suppose  $q = (0, 0)$ . Writing  $h(x) = h_k x^k + \dots$ , where  $k > 0$  and  $h_k \neq 0$ , we define the multiplicity of the fiber  $R$  as  $l(\mathcal{F}, R) = k$ . We want to prove the following property of  $\mathcal{F}$ :

**Lemma 4.1.** *The exceptional divisor of the reduction of singularities of  $\mathcal{F}$  at  $q = (0, 0)$  is a chain of rational curves  $L_1, \dots, L_n$  such that there is at most one non-invariant component, and if  $L_i$  is such component then*

$$L_i \cap L_j \neq \emptyset \Rightarrow \text{Sing}(\tilde{\mathcal{F}}) \cap L_j = 1 - \delta_{ij}$$

where  $\tilde{\mathcal{F}}$  is the reduced foliation and  $\delta_{ij}$  is the Kronecker's delta, that is,  $\delta_{ii} = 1$  and  $\delta_{ij} = 0$  if  $i \neq j$ .

*Proof.* If the linear part of  $\omega$  at  $q$  is non trivial, the result can be checked directly. We then suppose that the linear part at  $q$  is trivial. Then  $b(0) = c(0) = c'(0) = 0$  and  $l(\mathcal{F}, R) = k > 1$ . Since  $\text{Sing}(\omega) \subset \text{Sing}(\mathcal{F})$  has codimension two, we have  $a(0) \neq 0$ . Therefore  $\omega$  has algebraic multiplicity two at  $q$ . Since  $b(0)^2 - 4a(0)c(0) = 0$ ,  $q$  is the unique singularity of  $\mathcal{F}$  in  $R$ . The blow-up at  $q$  has on  $R' \cap E'$  ( $E'$  is the exceptional divisor and  $R'$  is the strict transform of  $R$ ) a singularity of the type  $d(xy) = 0$  and no more singularities on  $R'$ . If we collapse  $R'$ , then  $E'$  becomes a new fibre  $R_1$  of a new Riccati foliation  $\mathcal{F}_1$ . In this way, there may be at most two singularities on  $R_1$ , but now  $l(\mathcal{F}_1, R_1) < l(\mathcal{F}, R) = k$ .

Applying this procedure (*flipping of fibre*) a finite number of times, we obtain a foliation  $\mathcal{F}_m$  and an invariant fibre  $R_m$  such that a generating 1-form for the foliation has algebraic multiplicity one. That is, if  $\omega$  is that 1-form, then

$$\omega_m = (a_m(x)y^2 + b_m(x)y + c_m(x))dx + h_m(x)dy.$$

with  $c_m(0) = h_m(0) = 0$ , but  $b_m(0) \neq 0$  or  $c'_m(0) \neq 0$  or  $h'_m(0) \neq 0$ . Now, if the singularity  $(0,0)$  is dicritical, then the generating vector field for the foliation has two non zero linearly independent eigenvectors, and the exceptional divisor of the reduction of singularities  $\tilde{\mathcal{F}}_m$  at  $(0,0)$  is a chain of rational curves  $L_1, \dots, L_n$ , such that if  $L_i$  is the (unique) non-invariant component and  $L_i \cap L_j \neq \emptyset$  then  $\text{Sing}(\tilde{\mathcal{F}}_m) \cap L_j = 1 - \delta_{ij}$ . Since we can come back by blow-ups at points not equal to the  $(0,0)$  point of  $\mathcal{F}_m$  to the blow-up of the original foliation at the original singular point  $q = (0,0)$ , the property is also true for the reduction at  $q$  and then we conclude the proof.  $\square$

**Proposition 4.2.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$ . Let  $C = C_1 \cup \dots \cup C_n$  be a cycle of  $n$  invariant smooth rational curves, where  $n > 1$ . Suppose that  $C \cap \text{Sing}(\mathcal{F}) = \bigcup_{i \neq j} C_i \cap C_j$  are reduced non-degenerate singularities of  $\mathcal{F}$ . If  $\mathcal{F}$  is Riccati with respect to a rational fibration  $\pi : X \rightarrow B$ , then every fibre of  $\pi$  through a point of  $C \cap \text{Sing}(\mathcal{F})$  is completely supported on  $C$ .*

*Proof.* Let  $p \in C \cap \text{Sing}(\mathcal{F})$ . If  $R = \pi^{-1}(\pi(p))$  is the fibre through  $p$ , we can write

$$R = C_{i_1} \cup \dots \cup C_{i_k} \cup E_1 \cup \dots \cup E_l$$

where  $i_1, \dots, i_k \in \{1, \dots, n\}$  and  $E_1, \dots, E_l$  are smooth rational curves not in  $\{C_1, \dots, C_n\}$ , and, by Theorem 2.8 (see [1], page 192), there is a birational transformation

$$\sigma = \sigma_m \circ \dots \circ \sigma_1 : X \rightarrow Y$$

where each  $\sigma_i$ ,  $i = 1, \dots, m$ , is a blow-up at a point  $p_i$ , such that  $S = \sigma(R)$  is a regular fibre for the fibration  $\rho = \pi \circ \sigma^{-1}$  ( $\sigma$  is contraction of components of  $R$ ).

Note that if we blow-up a regular point of a foliation, the exceptional divisor is invariant, with only one singularity on it, of type  $xdy + ydx$ . Therefore if  $p_i$  is a regular point for the induced foliation  $(\sigma_m \circ \dots \circ \sigma_i)_* \mathcal{F}$ , then  $(\sigma_m \circ \dots \circ \sigma_i)^{-1}(p_i) = D_1 \cup \dots \cup D_r$  is  $\mathcal{F}$ -invariant and there exists  $D_l$  (rational curve) such that  $\#D_l \cap (D_1 \cup \dots \cup \widehat{D_l} \cup \dots \cup D_r) = \#D_l \cap \text{Sing}(\mathcal{F}) = 1$ . Now, if  $C \cap (\sigma_m \circ \dots \circ \sigma_i)^{-1}(p_i) \neq \emptyset$ , then, since  $(\sigma_m \circ \dots \circ \sigma_i)^{-1}(p_i)$  is connected and  $\mathcal{F}$ -invariant, we conclude that  $(\sigma_m \circ \dots \circ \sigma_i)^{-1}(p_i) \subset C$ , hence  $D_l = C_{i_l}$ , which result the contradiction  $1 = \#D_k \cap \text{Sing}(\mathcal{F}) = \#C_{i_l} \cap \text{Sing}(\mathcal{F}) = 2$ . Then, if we contract  $(\sigma_m \circ \dots \circ \sigma_i)^{-1}(p_i)$ , we don't affect the cycle  $C$ .

So we can look at  $\sigma$  as a reduction of singularities of  $\sigma_*(\mathcal{F})$  in  $S$  and use Lemma 4.1 to conclude: if  $p \in C_i \cap C_j$  then  $C_i$  or  $C_j$  is a component of  $R$ , otherwise we will have a non-invariant component of  $R$  with singularity.

If the set  $\{E_1, \dots, E_l\}$  is not empty, since  $R$  is connected, there exist  $C_i$  and  $E_j$  components of  $R$  such that  $C_i \cap E_j \neq \emptyset$ . Then  $E_j$  is not  $\mathcal{F}$ -invariant. But  $C_i$  has two singularities, then by Lemma 4.1  $C_i$  cannot intersect  $E_j$ . Then we have  $\{E_1, \dots, E_l\} = \emptyset$ .  $\square$

**Definition 4.3.** Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$ . A  $(k, l)$ -cycle for  $\mathcal{F}$  is a cycle of  $k > 1$  smooth rational curves  $C = C_1 \cup \dots \cup C_k \subset X$  such that:

- (1)  $C^2 = n > 0$ ;
- (2)  $C_i^2 = l$ ,  $i = 1, \dots, n$ ;
- (3)  $C$  is  $\mathcal{F}$ -invariant;

(4)  $C \cap \text{Sing}(\mathcal{F}) = \bigcup_{i \neq j} C_i \cap C_j$  are reduced nondegenerate singularities of  $\mathcal{F}$ .

**Corollary 4.4.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$  and let  $C = C_1 \cup \dots \cup C_k \subset X$  be a  $(k, l)$ -cycle for  $\mathcal{F}$ . Then  $(k, l) \in \{(2, -1), (3, -1), (3, 1), (6, -1)\} \cup \{(2m, 0) \mid m \in \mathbb{N}\}$ .*

*Proof.* The proof is just an easy application of Proposition 4.2, using suitable blow-ups at the crossing points of the cycle or blow-downs of exceptional curves.

Let  $C = C_1 \cup \dots \cup C_k \subset X$  be a  $(k, l)$ -cycle for a foliation  $\mathcal{F}$  on  $X$ . We can suppose that  $C_i \cap C_{i+1} = \{p_i\}$ ,  $i = 1, \dots, k-1$ , and  $C_k \cap C_1 = \{p_k\}$ , where the  $k$  points  $p_1, \dots, p_k$  are distinct.

If  $l > 0$ , choose  $z \in C$  a crossing point. After a suitable sequence of  $l$  blow-ups beginning at  $z$ , we obtain a new cycle of rational curves

$$\tilde{C} = E_l \cup \dots \cup E_1 \cup D_1 \cup D_2 \cup \dots \cup D_k$$

where  $D_1^2 = 0$ ,  $E_1^2 = -1$ ,  $E_2^2 = -2, \dots$ ,  $E_l^2 = -2$ ,  $D_2^2 = l$ ,  $D_3^2 = l, \dots$ ,  $D_k^2 = l-1$ , and  $D_1 \cap E_1 = \{p\}$ ,  $D_1 \cap D_2 = \{q\}$ . Then, the foliation  $\mathcal{F}$  is Riccati with respect to a rational fibration that has  $D_1$  as a regular fibre. By Proposition 4.2, a fibre  $R$  through a point not in  $D_1$  must be supported on  $\tilde{C}$ , and such a fibre must be also disjoint from  $D_1$ , since  $D_1$  is a fibre. That is, we must have  $R \subset \tilde{C} - D_1 \subset E_l \cup \dots \cup E_1 \cup D_2 \cup \dots \cup D_k$ . Since  $D_1 \cap E_1 \neq \emptyset$  and  $D_1 \cap D_2 \neq \emptyset$ ,  $R \subset \tilde{C} - (D_1 \cup E_1 \cup D_2) \subset E_l \cup \dots \cup E_2 \cup D_3 \cup \dots \cup D_k$ . If  $k = 2$  and  $l = 1$ , then, in fact,  $R \subset \tilde{C} - (D_1 \cup E_1 \cup D_2) = \emptyset$ , and we obtain a contradiction, since  $R$  cannot be empty. For  $k > 2$  or  $l > 1$ , every connected curve supported on  $E_l \cup \dots \cup E_2 \cup D_3 \cup \dots \cup D_k$  cannot be contracted to a rational curve of zero self-intersection, hence cannot be a fibre of a rational fibration. Therefore, there is no  $(k, l)$ -cycle if  $l > 0$ .

Now, suppose  $l = 0$ . Then, since  $C_i^2 = 0$ ,  $i = 1, \dots, k$ , we don't need take blow-ups to produce rational fibrations. Just choose, for example,  $C_1$  as the fibre  $R_1$  of a rational fibration and  $\mathcal{F}$  Riccati with respect to this fibration. Suppose that  $k = 2m + 1$  is odd. Take the fibre  $R_2$  through the crossing point  $p_3$ . Since  $R_2$  must be supported on  $C$ , we obtain  $R_2 = C_3$ . By the same reason, the fibre  $R_3$  through the crossing point  $p_5$  is  $R_3 = C_5$ . Inductively, we obtain that the fibre  $R_i$  through  $p_{2i-1}$  is  $R_i = C_{2i-1}$ . Then  $R_{m+1} = C_{2m+1} = C_k$  is the fibre through  $p_{2m+1} = p_k$ , which is impossible since the fibre through  $p_k = p_{2m+1}$  is just  $C_1 \neq C_k$ . Hence, if  $l = 0$ , then  $k$  must be even.

Finally, using contractions instead of blow-ups, we can conclude that there is no  $(k, -1)$ -cycle if  $(k, -1)$  is not in  $\{(2, -1), (3, -1), (6, -1)\}$ . □

We can now give here a different proof of [2, Chapter 3, Proposition 4].

**Proposition 4.5.** *Let  $\mathcal{F}$  be one of the foliations  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  or  $\mathcal{F}_3$ . Then  $\mathcal{F}$  is not birational to a Riccati foliation.*

*Proof.* Just like before, after one blow-up at the nodal point in the link of  $\mathcal{F}$ , we conclude, by Proposition 4.2, that  $\mathcal{F}$  cannot be Riccati. □

## 5. PROOF OF THE THEOREM 1.3

**5.1. Preliminary computations.** Let  $p$  be the node of  $C$  and  $C^2 = n$  a positive integer. If the hypotheses for the foliation are as in the Introduction 1 (that is,  $C$

is a link for  $\mathcal{F}$ ), we can use the Camacho-Sad formula to calculate the quotient of eigenvalues of  $\mathcal{F}$  at  $p$  (see [2, Chapter 3]):

$$n = C^2 = CS(\mathcal{F}, Y, p) = \lambda + 2 + \frac{1}{\lambda}.$$

Then we have the equation

$$\lambda^2 + (2 - n)\lambda + 1 = 0$$

whose solution is

$$\lambda = \frac{n - 2 \pm \sqrt{n(n - 4)}}{2}.$$

Therefore:

- (1) if  $C^2 = 1$  then  $-\lambda$  is a  $6^{th}$  primitive root of unit;
- (2) if  $C^2 = 2$  then  $-\lambda$  is a  $4^{th}$  primitive root of unit;
- (3) if  $C^2 = 3$  then  $-\lambda$  is a  $3^{th}$  primitive root of unit;
- (4) if  $C^2 = 4$  then  $\lambda = 1$ ;
- (5) if  $C^2 > 4$  then  $\lambda$  is a positive irrational number.

**5.2. Basic lemmas and propositions.** Here we will develop some more "technology" for the proof of our main result.

The next lemma is the generalization of [2, Chapter 3, Lemma 1]. The proof is essentially the same.

**Lemma 5.1.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$  and let  $C \subset X$  be a link for  $\mathcal{F}$  with node  $p \in C$ . Let  $L = N_{\mathcal{F}}^* \otimes \mathcal{O}_X(C)$  and  $\lambda$  be the quotient of eigenvalues at  $p$ . Suppose that  $-\lambda$  is a  $k^{th}$  primitive root of unit,  $k > 2$ . Then there exists a neighbourhood  $U$  of  $C$  such that  $L^{\otimes k}|_U = \mathcal{O}_U$ .*

*Proof.* Since  $\lambda$  is non-real, given a point  $q \in C - \{p\}$  and a transversal  $T$  to  $\mathcal{F}$  at  $q$ , the corresponding holonomy group of  $\mathcal{F}$ ,  $Hol_{\mathcal{F}} \subset \text{Diff}(T, q)$ , is infinite cyclic, generated by an hyperbolic diffeomorphism with linear part  $\exp(2\pi i \lambda)$  ([4] or [10]). Hence, there exists on  $T$  a  $Hol_{\mathcal{F}}$ -linearising coordinate  $z$ ,  $z(q) = 0$ . We extend this coordinate to a full neighbourhood of  $q$  in  $X$ , constantly on the local leaves of  $\mathcal{F}$ . The logarithmic 1-form  $\eta_q = \frac{dz}{z}$  defines  $\mathcal{F}$ , is closed, and  $\eta_q|_T$  is  $Hol_{\mathcal{F}}$ -invariant.

By the Poincaré linearisation theorem, in a neighbourhood of  $p$  the foliation is defined by a closed logarithmic 1-form  $\eta_p = \frac{dz}{z} - \lambda \frac{dw}{w}$  ([4] or [10]). If  $q$  is close to  $p$ , then  $\eta_p|_T$  is  $Hol_{\mathcal{F}}$ -invariant.

We obtain a neighbourhood  $U$  of  $C$  by the union of the open sets  $U_j$ , such that in each  $U_j$  the foliation is defined by a logarithmic 1-form  $\eta_j$ , with poles on  $C$ , which is closed and  $Hol_{\mathcal{F}}$ -invariant at the transversals. On  $U_i \cap U_j$  we have  $\eta_i = f_{ij}\eta_j$ ,  $f_{ij} \in \mathcal{O}^*$ . The closedness of  $\eta_i$  and  $\eta_j$  implies that  $df_{ij} \wedge \eta_j = 0$ , then  $f_{ij}$  is constant along the local leaves of  $\mathcal{F}$ . Moreover,  $f_{ij}|_T$  is  $Hol_{\mathcal{F}}$ -invariant and hence constant because the holonomy is hyperbolic.

Thinking  $\eta_j$  as local sections of  $L = N_{\mathcal{F}}^* \otimes \mathcal{O}_X(C)$ , then the previous property shows that  $L|_U$  is defined by a locally constant cocycle. Hence, to show that  $L^{\otimes k}|_U = \mathcal{O}_U$  it is sufficient to show that  $L^{\otimes k}|_C = \mathcal{O}_C$ . We can now use the residue of  $\eta_j$  along  $C$  to calculate the cocycle. For  $\eta_q$  with  $q \in C - \{p\}$  we can choose the 1-form to produce any non-zero residue. But we have a restriction around  $p$ :

the residue of  $\eta_p$  on one separatrix is  $-\lambda$  times the residue on the other separatrix. Since  $(-\lambda)^k = 1$ , it is clear that  $L^{\otimes k}|_C = \mathcal{O}_C$ .  $\square$

Also the next proposition is an easy adaptation of Brunella's argument in [2, Chapter 3, page 61-62].

**Proposition 5.2.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $X$  and let  $C \subset X$  be a link for  $\mathcal{F}$  with node  $p \in C$ . Let  $\lambda$  be the quotient of eigenvalues at  $p$ . Suppose that  $-\lambda$  is a  $k^{\text{th}}$  primitive root of unit,  $k > 2$ . Then there exists a compact surface  $Z$ , a transformation  $f : Z \rightarrow X$ , a neighbourhood  $U$  of  $C$  and an open set  $V \subset Z$  such that  $f|_V : V \rightarrow U$  is a regular  $k$ -covering over  $U$ . Moreover,  $f|_V^{-1}(C)$  is a cycle of  $k$  smooth rational curves, each one with self-intersection  $C^2 - 2$  (that is, a  $(k, C^2 - 2)$ -cycle), and the deck transformations of  $f|_V$  permutes cyclically the curves in the cycle.*

*Proof.* By the above lemma, the line bundle  $L^{\otimes k}$  has a nontrivial section over  $U$  without zeroes. Since  $C^2 > 0$ , the open set  $X - C$  is strictly pseudoconvex by Lemma 2.10. Then, by Lemma 2.9, that section can be extended to the full  $X$  as a global meromorphic section  $s$  of  $L^{\otimes k}$ . Consider  $E(L^{\otimes k})$  the compactification of the total space of  $L^{\otimes k}$ . Let  $\tilde{s}$  the compactification of the graph of  $s$  in  $E(L^{\otimes k})$ . Let  $\tau : E(L) \rightarrow E(L^{\otimes k})$  be the map defined by the  $k^{\text{th}}$  tensor power.

Let  $Z$  be the desingularisation of  $\tau^{-1}(\tilde{s})$  and elimination of indeterminacies of the projection  $\tau^{-1}(\tilde{s}) \dashrightarrow X$ . Take  $f : Z \rightarrow X$  the induced projection.  $\square$

**Lemma 5.3.** *Let  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (0 : 1 : 0)$ ,  $p_3 = (0 : 0 : 1)$  be three non collinear points in  $\mathbb{P}^2$ . Let  $\gamma \in \text{Aut}(\mathbb{P}^2)$  given by  $\gamma(z_1 : z_2 : z_3) = (z_3 : z_1 : z_2)$ . If  $J \in \text{Aut}(\mathbb{P}^2)$  is another automorphism such that  $J(p_1) = p_2$ ,  $J(p_2) = p_3$  and  $J(p_3) = p_1$ , then  $J$  is conjugated to  $\gamma$ , that is, there is  $g \in \text{Aut}(\mathbb{P}^2)$  such that  $\gamma = g \circ J \circ g^{-1}$ .*

*Proof.* In homogeneous coordinates,  $J(z_1 : z_2 : z_3) = (xz_3 : yz_1 : zz_2)$ , where  $xyz \neq 0$ . Note that we can suppose  $xyz = 1$ . Since  $\text{Aut}(\mathbb{P}^2) = \text{PGL}(3, \mathbb{C})$ , writing

$J$  and  $\gamma$  as matrices,  $J = \begin{pmatrix} 0 & 0 & x \\ y & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$  and  $\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ , we need to show

that there is a matrix  $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})$ , such that  $AJ = \gamma A$  in

$\text{PGL}(3, \mathbb{C})$ .

If  $a = (a_1, a_2, a_3)$ ,  $b = (b_1, b_2, b_3)$ ,  $c = (c_1, c_2, c_3)$ , it's easy to see that the equality  $AJ = \gamma A$  is equivalent to  $x\gamma(c) = a$ ,  $y\gamma(a) = b$ ,  $z\gamma(b) = c$ . Take  $a \neq 0$  and define  $b = y\gamma(a)$  and  $c = z\gamma(b) = zy\gamma^2(a)$ . Then the matrix  $A = (a, b, c) \in \text{GL}(3, \mathbb{C})$  is a solution.  $\square$

**Proposition 5.4.** *Let  $\mathcal{F}$  be a foliation on a compact complex surface  $Z$  and let  $C_1 \cup C_2 \cup C_3 \subset Z$  be a  $(3, 1)$ -cycle for  $\mathcal{F}$ . Suppose that there exists a birational  $\mathcal{F}$ -automorphism  $\phi : Z \dashrightarrow Z$  of order three permuting cyclically the rational curves. Then  $\mathcal{F}$  is birational to the linear foliation  $\mathcal{L}$  on  $\mathbb{P}^2$  from subsection 3.1 and the quotient foliation  $\mathcal{F}/\phi$  is birational to  $\mathcal{F}_3 = \mathcal{L}/\gamma$ .*



*Proof.* We can suppose  $\phi(C_1) = C_2$ ,  $\phi(C_2) = C_3$  and  $\phi(C_3) = C_1$ . Take, for each  $i$ , a section  $s_i$  of  $\mathcal{O}_Z(C_i)$  vanishing on  $C_i$ . Since  $C_1, C_2, C_3$  are linearly equivalent, we can define a rational map

$$(s_1 : s_2 : s_3) : Z \dashrightarrow \mathbb{P}^2.$$

It's easy to see that this map is birational and biregular in a neighbourhood of the cycle  $C_1 \cup C_2 \cup C_3$ , whose image is a cycle of three lines in  $\mathbb{P}^2$ . The induced foliation  $\tilde{\mathcal{F}}$  on  $\mathbb{P}^2$  is linear because the degree of the foliation is 1. The birational automorphism  $\phi$  is mapped to a birational automorphism  $\tilde{\phi}$  of  $\mathbb{P}^2$  which is biregular in a neighbourhood of the three lines and hence everywhere; moreover these automorphism permutes cyclically the three lines. By Lemma 5.3  $\tilde{\phi}$  is conjugated to the automorphism  $\gamma(z_1 : z_2 : z_3) = (z_3 : z_1 : z_2)$ , that is, there is  $g \in \text{Aut}(\mathbb{P}^2)$  such that  $\gamma = g \circ \tilde{\phi} \circ g^{-1}$ . Since  $\gamma$  is an  $g_*\tilde{\mathcal{F}}$ -automorphism, an easy computation shows that  $g_*\tilde{\mathcal{F}} = \mathcal{L}$  in homogeneous coordinates  $[z_1 : z_2 : z_3]$ . In particular,  $\mathcal{F}/\phi$  is birational to  $\mathcal{F}_3 = \mathcal{L}/\gamma$ .  $\square$

Analogously we can prove the following two results.

**Lemma 5.5.** *Let  $p_1 = (1 : 0, 1 : 0)$ ,  $p_2 = (0 : 1, 1 : 0)$ ,  $p_3 = (0 : 1, 0 : 1)$ ,  $p_4 = (1 : 0, 0 : 1)$  be four points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\beta \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  given by  $\beta(z_1 : z_2, z_3 : z_4) = (z_4 : z_3, z_1 : z_2)$ . If  $J \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  is another automorphism such that  $J(p_1) = p_2$ ,  $J(p_2) = p_3$ ,  $J(p_3) = p_4$  and  $J(p_4) = p_1$ , then  $J$  is conjugated to  $\beta$ , that is, there is  $g \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  such that  $\beta = g \circ J \circ g^{-1}$ .*  $\square$

**Proposition 5.6.** *Let  $\mathcal{H}$  be a foliation on a compact complex surface  $W$  and let  $D_1 \cup D_2 \cup D_3 \cup D_4 \subset W$  be a  $(4, 0)$ -cycle for  $\mathcal{H}$ . Suppose that there exists a birational  $\mathcal{H}$ -automorphism  $\phi : W \dashrightarrow W$  of order four permuting cyclically the rational curves. Then  $\mathcal{H}$  is birational to the linear foliation  $\mathcal{M}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  from subsection 3.2 and the quotient foliation  $\mathcal{W}/\phi$  is birational to  $\mathcal{F}_2 = \mathcal{M}/\beta$ .*

*Proof.* Take, for every  $i$ , a section  $s_i$  of  $\mathcal{O}_Z(D_i)$  vanishing on  $D_i$ . We define a rational map

$$(s_1 : s_2, s_3 : s_4) : W \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

It's easy to see that this map is birational and biregular in a neighbourhood of the cycle  $D_1 \cup D_2 \cup D_3 \cup D_4$ , whose image is a cycle of four lines in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Therefore, the induced foliation  $\tilde{\mathcal{H}}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  leaves invariant the cycle of four lines

$$(\mathbb{P}^1 \times \{0\}) \cup (\mathbb{P}^1 \times \{\infty\}) \cup (\{0\} \times \mathbb{P}^1) \cup (\{\infty\} \times \mathbb{P}^1)$$

whose singularities on the cycle are only the crossing points, each one reduced nondegenerate. According to [2, Chapter 4, Proposition 1] (see also [8] and [9]) we have that this foliation on  $\mathbb{P}^1 \times \mathbb{P}^1$  is given in affine coordinates  $(x, y)$  by a linear 1-form

$$\omega = \lambda y dx - x dy.$$

The birational automorphism  $\phi$  is mapped to a birational automorphism  $\tilde{\phi}$  of  $\mathbb{P}^1 \times \mathbb{P}^1$  which is biregular in a neighbourhood of the four lines and hence everywhere; moreover these automorphism permutes cyclically the four lines. By Lemma 5.5  $\tilde{\phi}$  is conjugated to the automorphism  $\beta(z_1 : z_2, z_3 : z_4) = (z_3 : z_4, z_2 : z_1)$ , that is, there is  $g \in \text{Aut}(\mathbb{P}^2)$  such that  $\beta = g \circ \tilde{\phi} \circ g^{-1}$ . Since  $\beta$  is an  $g_*\tilde{\mathcal{H}}$ -automorphism, an easy

computation shows that  $g_*\tilde{\mathcal{H}} = \mathcal{M}$  in homogeneous coordinates  $[z_1 : z_2, z_3 : z_4]$ . In particular,  $\mathcal{H}/\phi$  is birational to  $\mathcal{F}_2 = \mathcal{M}/\beta$ .  $\square$

Now we are ready to finish the proof of the theorem.

**5.3. Self-intersection 1.** Since  $-\lambda$  is a  $6^{th}$  primitive root of unit, by Proposition 5.2 we obtain a covering  $F : Z \rightarrow X$ , regular and of order six in a neighbourhood  $U$  of  $C$ . The deck transformations over  $U$  extend, by construction, to birational transformations of  $Z$ . Let  $\alpha : Z \dashrightarrow Z$  be the extended deck transformation of order six.

Now, we lift  $\mathcal{F}$  to  $Z$  via  $F$ , obtaining a new foliation  $\mathcal{G}$  which leaves invariant six smooth rational curves  $C_i$ ,  $i = 1, \dots, 6$ , forming a cycle over  $C$ . We have  $C_i^2 = -1$ , because  $C^2 = 1$ . The only singularities of  $\mathcal{G}$  at the cycle are the six crossing points, all reduced nondegenerate as well as  $p$ .

We can contract three disjoint  $(-1)$ -curves of the cycle, say  $C_1, C_3$  and  $C_5$ , obtaining a foliation  $(\tilde{\mathcal{G}}, \tilde{Z})$  birational to  $(\mathcal{G}, Z)$ . Note that  $\tilde{\mathcal{G}}$  has an invariant cycle of three smooth rational curves with self-intersection 1. Furthermore,  $\alpha^2 = \alpha \circ \alpha$  induces a birational  $\tilde{\mathcal{G}}$ -automorphism that permutes cyclically this cycle. Therefore, by Proposition 5.4,  $\tilde{\mathcal{G}}$  is birational to the linear foliation  $\mathcal{L}$  on  $\mathbb{P}^2$  given in subsection 3.1. In the same way, contracting the three disjoint  $(-1)$ -curves  $C_2, C_4$  and  $C_6$ , we also obtain a foliation birational to  $(\mathcal{L}, \mathbb{P}^2)$ . Then  $\alpha : Z \dashrightarrow Z$  induces a  $\mathcal{L}$ -automorphism  $\tilde{\alpha} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ . Since  $\tilde{\alpha}$  is unique up to conjugation (Lemma 5.4), the same is true for  $\alpha$ . Therefore  $\mathcal{F}$  is birational to the foliation  $\mathcal{F}_1$  from subsection 3.3.

**5.4. Self-intersection 2.** In this case,  $-\lambda$  is a  $4^{th}$  primitive root of unit. By Lemma 5.2 we have a covering  $G : W \rightarrow X$ , which is regular and of order 4 on a neighbourhood of  $C$ . Lifting  $\mathcal{F}$  to  $W$ , we obtain a foliation  $\mathcal{H}$  which leaves invariant four smooth rational curves  $D_i$ ,  $i = 1, \dots, 4$ , forming a cycle over  $C$ . Analogously,  $D_i^2 = 0$ , because  $C^2 = 1$ . The only singularities of  $\mathcal{H}$  at the cycle are the four crossing points, all reduced nondegenerate as well as  $p$ . Hence Proposition 5.6 implies that  $\mathcal{F}$  is birational to  $\mathcal{F}_2$ .

**5.5. Self-intersection 3.** This case is covered by Theorem 1.1. Anyway, the proof is just Lemma 5.2 plus Proposition 5.4.

**5.6. Self-intersection 4.** In this case,  $\lambda = 1$ , therefore  $p$  is a dicritical linerizable singularity (in particular, after a blow-up at  $p$ , the self-intersection of the strict transform of  $C$  is  $C^2 - 4 = 0$ , so we obtain a rational fibration over  $\mathbb{P}^1$ ) by [4] or [10]. But, since  $\lambda$  is rational positive, the foliation is not reduced nondegenerate at  $p$ , hence this case is not possible in our assumptions.

**5.7. Self-intersection greater than 4.** Since  $k > 4$  we have that  $\lambda$  is a positive irrational number, hence the singularity is non-dicritical linerizable.

After  $k$  suitable blow-ups the self-intersection of the strict transform of  $C$  is  $\tilde{C}^2 = C^2 - 4 - k + 1 = n - 3 - k$  (the first blow-up at  $p$  and the following blow-ups at one of the two singular points of the foliation in the strict transform of  $C$ ). Therefore, after  $n - 3$  blow-ups we obtain  $\tilde{C}^2 = 0$ . Let  $\sigma : \tilde{X} \rightarrow X$  be the transformation obtained by composing theses blow-ups,  $\tilde{C} = \sigma^*(C)$ ,  $E = \sigma^{-1}(p) = C_1 + \dots + C_{(n-3)}$ , where the  $C_i$  are rational curves, with  $C_1^2 = -1$  and  $C_j^2 = -2$  if  $j > 1$ , and  $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F})$ .

Since  $\mathcal{Z}(\tilde{\mathcal{F}}, \tilde{C}) = 2$ ,  $\tilde{\mathcal{F}}$  is a Riccati foliation with respect to a fibration  $\pi : \tilde{X} \rightarrow B$ , where  $B$  is a smooth curve (by [2, Chapter 4, Proposition 1]). We can suppose that the fibration has connected fibres. Since the exceptional divisor  $E$  is a union of smooth rational curves, the base  $B$  is a smooth rational curve.

Let  $q = C_1 \cap C_2$ , which is a singularity of the foliation, and  $R$  the fibre (possibly singular) through  $q$ . By Proposition 4.2,  $R$  must be supported on  $E$ , which is impossible, since  $E$  has negative definite matrix of intersection.  $\square$

## REFERENCES

1. W. BART, C. PETERS AND A. VAN DE VEN, *Compact complex surfaces*. Springer-Verlag, 2003.
2. M. BRUNELLA, *Birational geometry of foliations*. Monografias de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000. 138 pp.
3. M. BRUNELLA, *Minimal Models of Foliated Algebraic Surfaces*. Bull. Soc. math. France, **127**, (1999), 289–305.
4. C. CAMACHO, P. SAD, *Pontos singulares de equações diferenciais analíticas*. Monografias de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1987.
5. R. HARTSHORNE, *Algebraic Geometry*. Graduate Texts in Mathematics **52**, Springer (1998).
6. R. HARTSHORNE, *Ample Subvarieties of Algebraic Varieties*. Lecture Notes in Mathematics **156**, Springer-Verlag (1970).
7. R. LAZARSFELD, *Positivity in Algebraic Geometry*. University of Michigan (2001).
8. A. LINS-NETO, *Construction of singular holomorphic vector fields and foliations in dimension two*. J. Differential Geometry, **26** (1987) 1–31.
9. A. LINS-NETO, B. SCÁRDUA, *Introdução à Teoria das Folheações Algébricas Complexas*. Monografias de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2011.
10. J.-F. MATTEI, R. MOUSSU, *Holonomie et intégrales premières*. Ann. Sci., ENS, **13** (1980), 469–523.
11. A. NEEMAN, *Ueda Theory: Theorems and problems*. AMS Memoires, **415** (1989).
12. T. OHSAWA, *Vanishing theorems on complete Kahler manifolds*. Publ RIMS, Kyoto Univ., **20** (1984), 21–38.
13. J. V. PEREIRA, *Fibrations, divisors and transcendental leaves. With an appendix by Laurent Meersseman*. J. Algebraic Geom., **15** (2006), no. 1, 87–110.
14. J.V. PEREIRA, P.F SANCHEZ, *Transformation groups of holomorphic foliations*. Commun. Anal. Geom. **10**(5), 1115–1123 (2002).
15. O. SUZUKI, *Neighborhoods of a Compact Non-Singular Algebraic Curve Imbedded in a 2-Dimensional Complex Manifold*. Publ RIMS, Kyoto Univ., **11** (1975), 185–199.
16. T. UEDA, *Neighborhood of a Rational Curve with a Node*. Publ RIMS, Kyoto Univ., **27** (1991), 681–693.

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